§ 4.5. Substitution Method

Key points: ① Differential Notation: If $u=g(x)$, then $du=g'(x)\cdot dx$.

② u-Sub: $\int f(\underbrace{g(x)}_{u})\cdot \underbrace{g'(x)\cdot dx}_{du} \xlongequal[du=g'(x)dx]{u=g(x)} \int f(u)\cdot du$

• Goal of u-Substitution Method: By changing the variable x into u, we convert the integral into one of those five basic integrals which we can deal with.

• Basic integrals: $\int x^n dx = \frac{1}{n+1}x^{n+1}+C$, $\int \sin x\, dx = -\cos x + C$, $\int \cos x\, dx = \sin x + C$

$\int \sec^2 x\, dx = \tan x + C$, $\int \sec x \cdot \tan x\, dx = \sec x + C$.

① Differential Notation and Composition of functions.

eg.1. $u=x^2+1$, $\frac{du}{dx}=(x^2+1)'=2x \Rightarrow \boxed{du=2x\cdot dx.}$

eg.2. If $f(x)=\sqrt{x}$, $u=x^2+1$, then $f(u)=\sqrt{u}=\sqrt{x^2+1}$

② u-Sub method for indefinite integral

eg.3. Evaluate $\int \sqrt{u}\cdot du \xlongequal{n=\frac{1}{2}} \frac{1}{\frac{1}{2}+1}\cdot u^{\frac{1}{2}+1}+C = \boxed{\frac{2}{3}u^{\frac{3}{2}}+C}$

eg.4. $\int \sqrt{x^2+1}\cdot 2x\cdot dx$.

Set $u=x^2+1$, According to eg 1. $du=2x\cdot dx$.

$=\int \sqrt{u}\cdot du$.

Plug in u and du. By substituting u and du the integral of x turns into an integral of u with simpler form, which we evaluate in eg 3.

$=\frac{2}{3}u^{\frac{3}{2}}+C$

$=\frac{2}{3}(x^2+1)^{\frac{3}{2}}+C$.

Replace u by x^2+1 in the last step.

• The key of u-sub is to find the right substitution u.

More examples:

eg.5 Evaluate $\int x\cdot(2x^2+1)^3\cdot dx$. Hint: $u=2x^2+1$ is the ugly part.
(14 Final)

$u=2x^2+1$, $du=4x\cdot dx$ $\Rightarrow$ $\boxed{x\cdot dx=\frac{1}{4}\cdot du}$

$$=\int(2x^2+1)^3\cdot x\cdot dx$$

$$=\int u^3\cdot\frac{1}{4}\cdot du=\frac{1}{4}\cdot\frac{1}{3+1}\cdot u^{3+1}+C$$

$$=\frac{1}{16}\cdot u^4+C=\boxed{\frac{1}{16}(2x^2+1)^4+C}$$

e.g.6 Evaluate $\int\frac{\sec(\frac{x}{2})\cdot\tan(\frac{x}{2})}{\sqrt{\sec(\frac{x}{2})}}\cdot dx$. Hint: $u=\sec(\frac{x}{2})$ is the ugly part.
(14 Final)

$u=\sec(\frac{x}{2})$, $du=\sec(\frac{x}{2})\tan(\frac{x}{2})\cdot\frac{1}{2}\cdot dx$. chain rule for $(\sec \blacksquare)'$
$=\sec\blacksquare\cdot\tan\blacksquare$

$$\Rightarrow 2du=\sec(\tfrac{x}{2})\cdot\tan(\tfrac{x}{2})\cdot dx$$

$$=\int\frac{2du}{\sqrt{u}}=\int 2\cdot u^{-\frac{1}{2}}\,du$$

$$=2\cdot\frac{1}{-\frac{1}{2}+1}\cdot u^{-\frac{1}{2}+1}+C$$

$$=2\cdot 2\cdot u^{\frac{1}{2}}+C \;\xrightarrow{\text{back to } x}\; \boxed{4\left(\sec(\tfrac{x}{2})\right)^{\frac{1}{2}}+C}$$

• Linear substitution and more general form of the five basic integrals.

☆ $\int(ax+b)^n=\frac{1}{n+1}\cdot x^{n+1}+C$, $\int\sin(ax+b)dx=\frac{-1}{a}\cos(ax+b)+C$, $\int\cos(ax+b)dx=\frac{1}{a}\sin(ax+b)+C$

$\int\tan(ax+b)\cdot\sec(ax+b)\cdot dx=\frac{1}{a}\sec(ax+b)+C$, $\int\sec^2(ax+b)dx=\frac{1}{a}\tan(ax+b)+C$.

eg.7. $\int\sin(1-2x)\,dx \;\xrightarrow[du=-2dx]{u=1-2x}\; \int\sin u\cdot\frac{du}{-2}=\frac{1}{-2}(-\cos u)+C=\boxed{\frac{-1}{-2}\cos(1-2x)+C}$

$\frac{du}{-2}=dx$

③ u-sub for definite integral.

eg.8. Evaluate the definite integral $\int_0^2 \frac{1}{\sqrt{9-4x}}\,dx$.

Ugly part: $9-4x$. $u=9-4x$, $du=-4\cdot dx \Rightarrow dx=\frac{du}{-4}$

Caution: $\int_0^2$ 0, 2 are for x. They also change as we substitute $9-4x$ by u.

$$\int_{x=0}^{x=2} \xrightarrow{u=9-4x} \int_{u=9-4\cdot 0=9}^{u=9-4\cdot 2=1}, \quad \int_{u=9}^{u=1}$$

$$\int_0^2 \frac{1}{\sqrt{9-4x}}\,dx \overset{u=9-4x}{=\!=\!=} \int_9^1 \frac{1}{\sqrt{u}}\cdot\frac{du}{-4}$$ Use the flipping trick

$$= \int_1^9 \frac{1}{4}\cdot u^{-\frac{1}{2}}\,du = \frac{1}{4}\cdot\frac{1}{\frac{1}{2}}u^{\frac{1}{2}}\Big|_1^9$$

$$= \frac{1}{2}\sqrt{u}\Big|_1^9 = \boxed{\frac{1}{2}\sqrt{9}-\frac{1}{2}\sqrt{1}} = \boxed{1}$$

eg.9. Evaluate $\int_0^{\frac{\pi}{2}} 3\cdot\tan(\frac{x}{2})\cdot\sec^2(\frac{x}{2})\,dx$. Hint: $(\tan ■)' = \sec^2 ■$

$u=\tan(\frac{x}{2})$. $du=\frac{1}{2}\sec^2(\frac{x}{2})\,dx$. $\int_{x=0}^{x=\frac{\pi}{2}} \longrightarrow \int_{u=\tan 0=0}^{u=\tan(\frac{\pi}{4})=1}$

$2du=\sec^2(\frac{x}{2})dx$.

$$= \int_0^1 3\cdot u\cdot 2du = 6\cdot\frac{1}{2}u^2\Big|_0^1 = \boxed{3\cdot 1^2 - 3\cdot 0^2 = 3}$$

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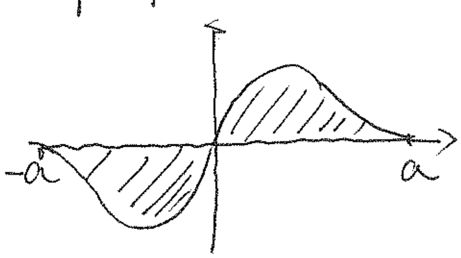
④ Symmetry integral.

If $f(x)$ is odd, i.e., $f(-x)=-f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$



f odd means the graph of f is symmetric about the origin. Then the area above and below x-axis are the same, which will be cancelled out.

eg.10. $f(x)=\sin x\cdot(x^2+1)$. $\int_{-8}^{8}\sin x\cdot(x^2+1)\,dx = 0$

since $f(-x)=\sin(-x)\cdot((-x)^2+1) = -\sin x\cdot(x^2+1) = -f(x)$ ($\sin x$ is odd)

More examples and hints for webwork:

eg 11. (ww8). (14 Final) $\int_8^{11} x\cdot\sqrt{x-7}\,dx$. Hint: $u=x-7$. $du=dx$.

$= \int x\cdot\sqrt{u}\,du$ there is still one x left. keep substituting via the relation $u=x-7 \Leftrightarrow u+7=x$.

$= \int_1^4 (u+7)\cdot\sqrt{u}\cdot du$ $\quad x=11 \to u=x-7=4$

$= \int_1^4 u\cdot u^{\frac{1}{2}} + 7\cdot u^{\frac{1}{2}}\,du$ $\quad x=8 \to u=x-7=1$

$= \int_1^4 u^{\frac{3}{2}} + 7\cdot u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} + 7\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^4$

$4^{\frac{5}{2}} = (\sqrt{4})^5 = 32.$

$4^{\frac{3}{2}} = (\sqrt{4})^3 = 8$

$= \frac{2}{5}\cdot 4^{\frac{5}{2}} + \frac{14}{3}\cdot 4^{\frac{3}{2}} - \left(\frac{2}{5}\cdot 1 + \frac{14}{3}\cdot 1\right)$

$= \frac{64}{5} + \frac{112}{3} - \frac{2}{5} - \frac{14}{3} = \boxed{\frac{62}{5} + \frac{98}{5}}$

eg 12. (ww6) $\int \frac{1}{x^2}\sin\left(\frac{3}{x}\right)\cdot\cos\left(\frac{3}{x}\right)dx$. Hint:

$u = \cos\left(\frac{3}{x}\right)$.

$du = -\sin\left(\frac{3}{x}\right)\cdot\left(\frac{3}{x}\right)'\,dx$. chain rule.

$= -\sin\left(\frac{3}{x}\right)\cdot\frac{-3}{x^2}\,dx$ $\quad$ inner $\frac{3}{x} = 3x^{-1}$

$= \sin\left(\frac{3}{x}\right)\cdot\frac{3}{x^2}\,dx$. $\quad (3\cdot x^{-1})' = 3\cdot\frac{-1}{x^2}$

$\Rightarrow \frac{1}{3}du = \sin\left(\frac{3}{x}\right)\cdot\frac{1}{x^2}\,dx$

$= \int \cos\left(\frac{3}{x}\right)\cdot\frac{1}{x^2}\sin\left(\frac{3}{x}\right)dx$.

$= \int u\cdot\frac{1}{3}du$

$= \frac{1}{3}\cdot\frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{6}\cdot\left(\cos\left(\frac{3}{x}\right)\right)^2 + C}$

eg 13. (14 Final) $\int \frac{x^3}{\sqrt{1+2\cdot x^4}}\,dx$. Ugly part: $u = 1+2x^4$, $du = 8\cdot x^3\,dx$

$\Rightarrow \frac{1}{8}du = x^3\,dx$

$= \int \frac{1}{\sqrt{1+2x^4}}\cdot x^3\,dx$

$= \int \frac{1}{\sqrt{u}}\cdot\frac{1}{8}du = \int u^{-\frac{1}{2}}\cdot\frac{1}{8}du = \frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1}\cdot\frac{1}{8} + C$

$= 2(u)^{\frac{1}{2}}\cdot\frac{1}{8} + C$

$= \frac{1}{4}\cdot(1+2x^4)^{\frac{1}{2}} + C$.